Advanced Short-Term Actuarial Mathematics Exam

Exam ASTAM

Date: April 23, 2025

INSTRUCTIONS TO CANDIDATES

General Instructions

1. This examination has 6 questions numbered 1 through 6 with a total of 60 points. The points for each question are indicated at the beginning of the question.

2. **Question 1 is to be answered in the Excel workbook. For this question only the work in the Excel workbook will be graded.**

3. Questions 2-6 are to be answered in pen in the Yellow Answer Booklet provided. For these questions graders will only look at the work in the Yellow Answer Booklet. Excel may be used for calculations or for statistical functions, but any work in the Excel booklet will not be graded.

4. While every attempt is made to avoid defective questions, sometimes they do occur. If you believe a question is defective, the supervisor or proctor cannot give you any guidance beyond the instructions provided in this document.

Excel Answer Instructions

1. For Question 1, you should answer directly in the Excel Question worksheet. The question will indicate where to record your answers.
2. You should generally use formulas in Excel rather than entering solutions as hard coded numbers. This will aid graders in assigning appropriate credit for your work.
3. Graders for Question 1 will not have access to any comments or calculations provided in the Yellow Answer Booklet.
4. For Question 1, you may add notes to the Excel Question worksheet if you feel that might help graders. However, these should be entered directly into the Excel Question worksheet. Graders may not be able to read notes entered as comments.
5. When you finish, save your Excel workbook with a filename in the format xxxxx_ASTAM where xxxxx is your candidate number. Your name must not appear in the filename.
6. Record your candidate number in the indicated cell in the Excel Question worksheet.

Pen and Paper Answer Instructions

1. Write your candidate number and the number of the question you are answering at the top of each sheet. Your name must not appear.

2. Start each question on a fresh sheet. You do not need to start each sub-part of a question on a new sheet.

3. Write in pen on the lined side of the answer sheet.

4. The answer should be confined to the question as set.

5. When you are asked to calculate, show all your work including any applicable formulas in the Yellow Answer Booklet.

6. If you use Excel for calculations for pen and paper answers, you should include as much information in the Yellow Answer Booklet as if you had used a calculator, including formulas and intermediate calculations where relevant. Written answers without sufficient support may not receive full or partial credit.

7. When you finish, hand in <u>all</u> your written answer sheets to the Prometric Center staff. Be sure to hand in all your answer sheets because they cannot be accepted later.

BEGINNING OF EXAMINATION
ADVANCED SHORT-TERM ACTUARIAL MATHEMATICS

Provide the response for Question 1 in the Excel Question worksheet

1.

(*10 points*) An insurance company has 10,000 policies in force during 2024. The information on the number of claims is given in Table 1. You actually have the exact data on claim frequency for policies with 3 or more claims, but for the purposes of the Chi-squared test they are grouped.

Table 1	
Number of Claims	Number of Policies
0	2400
1	3400
2	2400
3+	1800
Total	10000

Based on the more detailed information, you know that the average number of claims per policy is 1.48.

(a) (*3 points*) You want to test the following hypothesis using the Chi-squared test:

H0: The data are from a Poisson Distribution
H1: The data are not from a Poisson Distribution

(i) Calculate the Chi-squared test statistic.

(ii) State the degrees of freedom for this test.

(iii) Determine the *p*-value for this test.

(iv) State with reasons the conclusion of the Chi-squared test.

1. Continued

(b) (*3 points*) The company wants to model the claim severity. The sample of claims used to analyze the distribution is given in Table 2.

Table 2					
i	x_i	i	x_i	i	x_i
1	811	11	15831	21	43860
2	1195	12	17086	22	44501
3	2292	13	17894	23	46824
4	2892	14	22653	24	48515
5	4230	15	23384	25	57355
6	5054	16	24741	26	59263
7	5317	17	25166	27	67045
8	5662	18	27356	28	78154
9	8182	19	37717	29	112768
10	8222	20	38294	30	113035

The company wants to test the following hypothesis using the Kolmogorov-Smirnov Test:

H0: The data are from an exponential distribution with $\theta = 25,000$
H1: The data are not from an exponential distribution with $\theta = 25,000$

(i) Calculate the Kolmogorov-Smirnov test statistic.

(ii) State with reasons the conclusion of the Kolmogorov-Smirnov test.

(c) (*4 points*) You are given that the maximum likelihood estimate (MLE) of θ for the exponential distribution is the mean of the sample. That is $\hat{\theta} = \bar{x}$.

(i) Calculate $\hat{\theta}$.

(ii) Determine the standard deviation of $\hat{\theta}$.

(iii) Calculate the MLE of the probability that an individual claim is greater than 10,000.

(iv) Using the delta method, calculate the approximate standard deviation of the probability that an individual claim is greater than 10,000.

2.
(*11 points*)

(a) (*1 point*) Assume $N_1 \sim$ Poisson(λ_1) and $N_2 \sim$ Poisson(λ_2), and that N_1 and N_2 are independent.

Use probability generating functions to show that $N_1 + N_2 \sim$ Poisson$(\lambda_1 + \lambda_2)$.

An insurance company is analyzing a portfolio of 10 independent auto insurance policies. Drivers are classified into two categories, "Urban Drivers" and "Rural Drivers".

You are given:

(i) The annual number of claims from an "Urban Driver" follows a Poisson distribution with mean 0.5, and the annual number of claims from a "Rural Driver" follows a Poisson distribution with mean 0.3.

(ii) Four policies are issued to "Urban Drivers" and six policies are issued to "Rural Drivers".

(iii) The following table shows the distribution of claims severity, *X*, for "Urban Drivers" and "Rural Drivers":

Amount x	$\Pr[X = x]$ "Urban Drivers"	$\Pr[X = x]$ "Rural Drivers"
1	0.60	0.10
2	0.30	0.40
3	0.10	0.50

(b) (*2 points*) Calculate the mean and standard deviation of aggregate annual claims for an individual "Urban Driver" policy.

You are given that the aggregate annual claims for the portfolio, *S*, has a compound Poisson distribution with Poisson parameter λ and associated single claim distribution F_X. Let f_X denote the probability function of F_X.

(c) (*3 points*)

(i) Calculate λ.

(ii) Show that $f_X(1) = 0.36$ to the nearest 0.01. You should calculate the value to the nearest 0.001.

(iii) Calculate $f_X(2)$ and $f_X(3)$.

2. **Continued**

(d) (*3 points*)

(i) Calculate $\Pr[S=0]$.

(ii) Calculate $\Pr[S=2]$ using recursion or otherwise.

(e) (*2 points*) Calculate the net stop-loss premium for the total annual claims for the portfolio with an aggregate deductible of 3.

3.

(*10 points*) The ratemaking actuary for a property and casualty insurance company is analyzing automobile claims data.

You are given the following information about the portfolio:

(i) Policies are classified into three classes, A, B, and C, depending on the deductible chosen.

(ii) The annual number of claims per policy within each class is assumed to follow a Poisson distribution, with parameter λ varying across classes.

(iii) The ground-up loss severity for all classes is assumed to follow a Pareto distribution with $\alpha = 2$ and $\theta = 2500$.

(iv) Claim frequency and loss severity are independent.

(v) The current differentials are based on expected aggregate claim amount.

(vi) The table below shows the claim frequency information for the different classes.

Class	**Deductible**	**Claim Frequency Poisson, λ**
A (Base Rate)	500	0.0625
B	1000	0.0510
C	1500	0.0430

(a) (*1 point*) State two considerations regarding the data to use for determining deductible relativities.

(b) (*3 points*)

(i) Show that the expected aggregate claim amount for a Class A policy is 190 to the nearest 10. You should calculate the value to the nearest 0.1.

(ii) Calculate the current differential for Class B.

(iii) Calculate the current differential for Class C.

3. **Continued**

(c) (*3 points*) The current differentials are updated using the experience shown in the following table:

Class	Claims	Earned Exposure
A (Base Rate)	3,465,000	18,500
B	1,335,000	8,000
C	350,000	2,200

(i) Calculate the new indicated differential for Class B using the loss cost method.

(ii) Calculate the new indicated differential for Class C using the loss cost method.

(iii) Show that the off-balance factor using the new indicated differentials is 0.98 to the nearest 0.01. You should calculate the value to the nearest 0.001.

(d) (*3 points*) The current base rate for Class A is 200. The insurer decides to implement an overall rate increase of 10%.

(i) Show that the resulting percentage increase in the base rate is between 12% and 13%. You should calculate the value to the nearest 0.1%.

(ii) Show that the percentage increase in premium for Class C policies is between 4% and 5%. You should calculate the value to the nearest 0.1%.

(iii) The marketing manager suggests instead applying a uniform 10% increase to the premiums of all classes. Critique this suggestion.

4.

(*9 points*) An actuary is analyzing the reported losses from an auto insurance with an ordinary deductible of d. Let X denote the ground-up loss random variable, which is assumed to have a Pareto distribution with known θ and unknown α.

A random sample of 20 reported ground-up losses was collected. Note that losses below the deductible are not reported to the insurer, so only losses above the deductible were sampled.

(a) (*3 points*)

(i) Determine the log-likelihood function for the sample in terms of θ, α, d, and the sample data, $x_1, x_2, \cdots, x_{20}$.

(ii) You are given that $\theta = 5{,}000$, $d = 1{,}000$, and that $\sum_{i=1}^{20} \ln(\theta + x_i) = 181.66$.

Show that the maximum likelihood estimate of α, denoted $\hat{\alpha}$, is 2.6 to the nearest 0.1. You should calculate the value to the nearest 0.001.

(b) (*1 point*) Estimate a 95% confidence interval for α using Fisher's information.

(c) (*2 points*) Let $Y = \begin{cases} 0 & \text{if } X \le d \\ X - d & \text{if } X > d \end{cases}$

Calculate the maximum likelihood estimate of $E[Y]$.

(d) (*2 points*) For X with a Pareto distribution with $\theta = 5000$ and $\alpha = \hat{\alpha}$, show that the loss elimination ratio for an ordinary deductible of $d = 1000$ is 0.25 to the nearest 0.05. You should calculate the value to the nearest 0.001.

(e) (*1 point*) The actuary's colleague notes that an ordinary deductible of $d = 1000$ produces a similar reduction in expected payment as does a 75% coinsurance.

Explain why an insurer may prefer to impose an ordinary deductible of 1000 over 75% coinsurance.

5.

(*11 points*) You are calculating outstanding claim reserves for your company's portfolio of short-term insurance contracts.

You are given the following cumulative claims run-off triangle.

	Development Year, j			
Accident Year i	0	1	2	3
0	660	1155	1403	1568
1	792	1304	1650	
2	825	1568		
3	941			

(a) (*2 points*) Show that the estimated outstanding claims for Accident Year (AY) 3, using the chain ladder method, are 1370 to the nearest 10. You should calculate the value to the nearest 1.

You are given the following notation and assumptions for the Bornhuetter-Ferguson (BF) method.

Given parameters $\mu_i, i = 0,1,\cdots,I$ and $\beta_j,\ j = 0,1,2,\cdots,J$, where $\beta_J = 1$:

BF Assumption (1):

$$\mathrm{E}\left[C_{i,0}\right] = \beta_0\, \mu_i$$
$$\mathrm{E}\left[C_{i,j+1} \middle| C_{i,0}, C_{i,1}, \cdots, C_{i,j}\right] = C_{i,j} + \left(\beta_{j+1} - \beta_j\right)\mu_i$$

BF Assumption (2): $C_{i,j}$ and $C_{l,k}$ are independent for $i \neq l$, and for all $j,\ k$.

(b) (*3 points*)

(i) Show using iteration that $\mathrm{E}\left[C_{i,J} \middle| C_{i,j}\right] = C_{i,j} + \left(1-\beta_j\right)\mu_i$.

(ii) Show that $\mathrm{E}\left[C_{i,j}\right] = \beta_j\, \mu_i$.

(Hint: Use induction.)

5. **Continued**

You are given the following additional information.

(i) The estimated values of the β_j parameters are $\hat{\beta}_j = \dfrac{1}{\hat{\lambda}_j}$.

(ii) The BF estimate of ultimate cumulative claims from AY i, given $C_{i,j}$, is $\tilde{C}_{i,J} = C_{i,j} + (1 - \hat{\beta}_j)\mu_i$.

(iii) $\hat{C}_{i,J}$ is the chain ladder estimate of ultimate cumulative claims for AY i.

(c) (*3 points*)

(i) Derive the following formula for the estimated cumulative claims using the BF method:

$$\tilde{C}_{i,J} = \hat{\beta}_j \, \hat{C}_{i,J} + (1 - \hat{\beta}_j)\mu_i$$

(ii) Explain how the BF estimate of cumulative claims can be interpreted as a credibility estimate.

You are also given:

(i) For each AY i, μ_i is based on a loss ratio of 90%.

(ii) The earned premiums for AY 3 are 2400.

(d) (*2 points*) Calculate the BF estimate of the outstanding claims for AY 3.

(e) (*1 point*) Write down one advantage and one disadvantage of using the BF method over the chain ladder method.

6.

(*9 points*) For a portfolio of 100 independent policies, the annual claim amount on a policy, conditional on the risk parameter $\Theta = \theta$, has an exponential distribution with mean θ.

An actuary assumes that the prior distribution of Θ is an Inverse Gamma distribution, denoted $IG(\alpha, \beta)$, with density function:

$$\pi(\theta) = \frac{\beta^{\alpha} e^{-\left(\frac{\beta}{\theta}\right)}}{\theta^{\alpha+1}\Gamma(\alpha)}, \quad \theta, \alpha, \beta > 0$$

You are given that $\mathrm{E}\left[\Theta^k\right] = \dfrac{\beta^k \, \Gamma(\alpha - k)}{\Gamma(\alpha)}, k < \alpha.$

Let x_i denote the observed claim amount from the i^{th} policy, $i = 1, \ldots, 100$, in Year 1. The actuary first uses the Bayesian approach.

(a) (*2 points*) Show that the posterior distribution of Θ is $IG\left(\alpha + 100, \beta + \sum_{i=1}^{100} x_i\right)$.

(b) (*1 point*) You are given that $\alpha = 30, \beta = 10{,}000,$ and $\sum_{i=1}^{100} x_i = 50{,}000$.

Calculate the Bayesian estimate of the annual claim amount of a randomly selected policy in Year 2.

The actuary decides to approximate the Bayesian premium using the Bühlmann credibility model.

(c) (*3 points*)

(i) Show that the expected value of the hypothetical means, μ, is 350 to the nearest 50. You should calculate the value to the nearest 1.

(ii) Show that the expected value of the process variance, v, is 123,150 to the nearest 50. You should calculate the value to the nearest 1.

(iii) Show that the variance of the hypothetical means, a, is 4,250 to the nearest 50. You should calculate the value to the nearest 1.

6. **Continued**

(d) (*2 points*) For estimating the annual claim amount of a randomly selected policy from the portfolio in Year 2:

(i) Calculate the Bühlmann credibility factor.

(ii) Determine the Bühlmann credibility estimate.

(e) (*1 point*) Explain why the Bayesian and Bühlmann estimates are the same in this case.

****END OF EXAMINATION****