QFI Quantitative Finance Exam

Exam QFIQF

Date: Thursday, April 24, 2025

INSTRUCTIONS TO CANDIDATES

General Instructions

1. This examination has 10 questions numbered 1 through 10 with a total of 70 points.

   The points for each question are indicated at the beginning of the question.

2. While every attempt is made to avoid defective questions, sometimes they do occur. If you believe a question is defective, the supervisor or proctor cannot give you any guidance beyond the instructions provided in this document.

3. Each question part or subpart should be answered either in the Excel document or the paper provided as directed. Graders will only look at the work as indicated.

4. In the Excel document, answers should be entered in the box marked ANSWER. The box will expand as lines of text are added. There is no need to use special characters or subscripts (though they may be used). For example, β_1 can be typed as beta_1 (and ^ used to indicate a superscript).

5. Prior to uploading your Word and Excel files, each file should be saved and renamed with your unique candidate number in the filename. To maintain anonymity, please refrain from using your name and instead use your candidate number.

6. The Excel file that contain your answers must be uploaded before the five-minute upload period expires.

Written-Answer Instructions

1. Write your candidate number at the top of each sheet. Your name must not appear.

2. Write on only one side of a sheet. Start each question on a fresh sheet. On each sheet, write the number of the question that you are answering. Do not answer more than one question on a single sheet.

3. The answer should be confined to the question as set.

4. When you are asked to calculate, show all your work including any applicable formulas. When you are asked to recommend, provide proper justification supporting your recommendation.

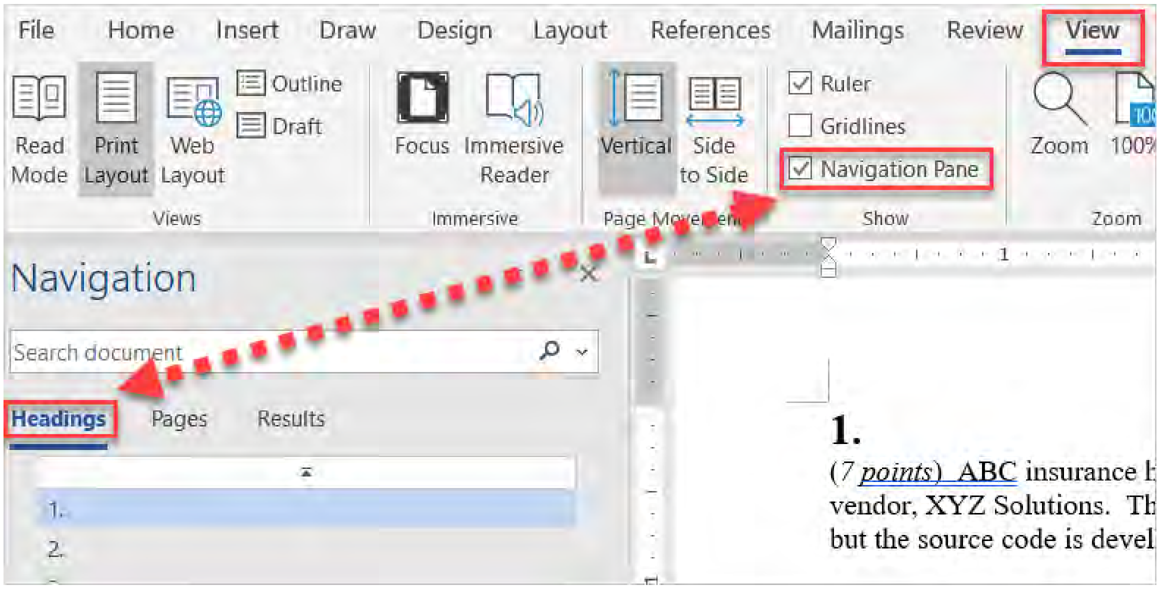
5. When you finish, hand in all your written-answer sheets to the Prometric center staff. Be sure to hand in all your answer sheets because they cannot be accepted later.

8770 W. Bryn Mawr Avenue, Suite 1000
Chicago, IL 60631

Navigation Instructions

Open the Navigation Pane to jump to questions.

Press Ctrl+F, or click View > Navigation Pane:



The responses for all parts of this question are required on the paper provided to you.

1.

(*5 points*) *X(t)* and *Y(t)* are solutions of stochastic differential equations shown below driven by the same Brownian motion *W(t)*:

$$dX(t) = \alpha(t)dt + \sigma(t)dW(t)$$
$$dY(t) = \mu(t)dt + v(t)dW(t)$$

(a) (*1 point*) Verify using Ito's Lemma that

$$d\left[\left(X(t) + Y(t)\right)^2\right]$$
$$= 2X(t)dX(t) + 2Y(t)dY(t) + 2X(t)dY(t) + 2Y(t)dX(t) + \left(\sigma(t) + v(t)\right)^2 dt$$

(b) (*1.5 points*) Verify, using Ito's Lemma and part (a), that

$$d\left(X(t)Y(t)\right) = X(t)dY(t) + Y(t)dX(t) + \sigma(t)v(t)dt$$

An asset has a price $S(t)$ which satisfies the following stochastic differential equation:

$$dS(t) = -10S(t)dt + 8dW(t)$$

The solution for *S(t)* is of the form:

$$S(t) = e^{-At}\left[B + C\int_0^t e^{Ds}\, dW(s)\right]$$

where *A*, *B*, *C*, and *D* are constants, *S(0)* = 1

(c) (*2.5 points*) Derive *A*, *B*, *C*, and *D* using part (b).

The responses for all parts of this question are required on the paper provided to you.

2.

(*6 points*) Suppose that $\{B_t, t \geq 0\}$ is a standard Brownian motion with $B_0 = 0$, and filtration $\{F_t, t \geq 0\}$.

(a) (*1.5 points*) Verify that $Cov\,(B_s, B_t) = \min\{s, t\}$.

(b) (*1 point*) Derive $E[B_4 - B_1 | F_3]$.

(c) (*2.5 points*) Derive the distribution of the Riemann integral $\int_0^1 e^s B_s ds$.

Now suppose that $\{W_t, t \geq 0\}$ is another standard Brownian motion adapted to the same filtration and is independent of $\{B_t, t \geq 0\}$. Define the process $\{X_t, t \geq 0\}$ by setting $X_t = B_t W_t,\ t \geq 0$.

(d) (*1 point*) Verify that $\{X_t, t \geq 0\}$ is not a standard Brownian motion.

The responses for all parts of this question are required on the paper provided to you.

3.

(*8 points*) Let the stock price process $S_t, 0 < t < T$, be given by the following stochastic differential equation, with the continuously compounded interest rate r and volatility σ both constants:

$$dS_t = r\, S_t dt + \sigma\, S_t dW$$

Consider a special European option with the following payoff:

$$P(S_T) = \begin{cases} 0 & \text{if } S_T \leq K - a \\ \left(\frac{1}{4a}\right)(S_T - K + a)^2 & \text{if } K - a < S_T \leq K + a \\ S_T - K & \text{if } S_T > K + a \end{cases}$$

where K is a strike price and a is a positive constant less than K.

(a) (*1.5 points*)

(i) Graph $P(S_T)$ as a function of S_T.

(ii) Compare $P(S_T)$ with the payoff of a vanilla European call option with the same underlying asset and strike price K.

(b) (*1 point*) Verify that $E_t\left[S_T^{\beta}\right] = S_t^{\beta} e^{\beta\left(r+\frac{1}{2}\sigma^2(\beta-1)\right)\tau}$, where $\tau = T - t$ and β is a constant.

(c) (*1 point*) Verify that for any $A > 0$

$$E_t\left[\mathbb{I}_{S_T > A}\right] = N\left(d_-\left(\frac{S_t}{A}, \tau\right)\right)$$

where $d_-(x,\tau) = \frac{\ln(x) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}$.

(d) (*0.5 points*) Verify that

$$E_t\left[\mathbb{I}_{K-a < S_T \leq K+a}\right] = N\left(d_-\left(\frac{S_t}{K-a}, \tau\right)\right) - N\left(d_-\left(\frac{S_t}{K+a}, \tau\right)\right)$$

3. **Continued**

(e) (*2 points*) Verify that for any $A > 0$,

$$E_t\left[S_T^\beta \mathbb{I}_{S_T>A}\right] = S_t^\beta e^{\beta\left(r+\frac{1}{2}\sigma^2(\beta-1)\right)\tau} N\left(d_-\left(\frac{S_t}{A},\tau\right)+\beta\sigma\sqrt{\tau}\right).$$

(f) (*2 points*) Verify that the no-arbitrage pricing formula for this option $C(t,S;K,a)$ is

$$C(t,S;K,a) = e^{-r\tau}\left(\frac{1}{4a}E_t[{S_T}^2\,\mathbb{I}_{K-a<S_T\le K+a}] - \frac{(K-a)}{2a}E_t\left[S_T\,\mathbb{I}_{K-a<S_T\le K+a}\right] + \frac{(K-a)^2}{4a}E_t\left[\,\mathbb{I}_{\,K-a<S_T\le K+a}\right] + E_t\left[S_T\,\mathbb{I}_{S_T>K+a}\right] - KE_t[\mathbb{I}_{\{S_T>K+a)}]\right)$$

The responses for all parts of this question are required on the paper provided to you.

4.

(*6 points*) The short rate r follows the SDE:

$$dr_t = \sigma dW_t$$

where $\{W_t : t \geq 0\}$ is a Wiener process and σ is a constant.

The zero-coupon bond price depends on this short rate and time to maturity T:

$$V(t,r,T) = E\left(exp\left(-\int_t^T r_s\, ds\right) \mid r_t = r\right)$$

(a) (*2.5 points*) Derive the bond price formula by finding $A(t,T)$ and $B(t,T)$ in the expression $V(t,r,T) = \exp(A(t,T) - B(t,T)\, r_t)$.

Hint: You could use Fubini's theorem in calculus such that

$$\int_S \left(\int_Q f(t,s) dW_t\right) ds = \int_Q \left(\int_S f(t,s) ds\right) dW_t$$

where $f(t,s)$ is an integrable function, $\{W_t : t \geq 0\}$ is a Wiener process and S, Q are time domains.

(b) (*1 point*) Verify that the bond pricing function $V(t,r,T)$ from part (a) satisfies the fundamental pricing PDE:

$$rV = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial r}\mu(r,t) + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}\sigma^2.$$

(c) (*2.5 points*) Derive the mean and the variance of r_T under T-forward risk-neutral measure.

The responses for some parts of this question are required on the paper provided to you.

5.

(*7 points*) You are working with the Cox, Ingersoll, and Ross (CIR) model for pricing some coupon bonds. You have calibrated the CIR model using two sets of data.

- Based on daily overnight rates from January 1, 2014 to December 31, 2024: calibrated parameters are ($\bar{r} = 0.0513, \gamma = 0.67,$ and $\alpha = 0.049$)
- Based on 25 STRIPS on December 31, 2024: calibrated parameters ($\bar{r} = 0.0624, \gamma = 0.57,$ and $\alpha = 0.049$)
- The continuously compounded overnight rate on December 31, 2024 is 5.04%

(a) (*1 point*) Identify the real-world model parameters and risk-neutral world parameters. Justify your choice.

(b) (*1.5 points*) Calculate the stationary mean and variance of the model in the real world.

Under the CIR model the probability density function (pdf) of $r_t|r_0$, $f(r_t\,|r_0)$, takes the following form:

$$f(r_t|r_0) = c_t\chi^2(c_t r_t, v, \lambda_t)$$

where $\chi^2(.\,, v, \lambda_t)$ is the pdf of the non-central chi squared distribution with v degrees of freedom, and non-centrality parameter λ_t, with

$$c_t = \frac{4\gamma}{\alpha(1 - \exp(-\gamma t))}$$
$$v = \frac{4\gamma\bar{r}}{\alpha}$$
$$\lambda_t = c_t r_0 \exp(-\gamma t)$$

for some positive α and γ.

(c) (*1.5 points*) Identify the stationary distribution.

(d) (*3 points*) Calculate the price of a 10-year 5% semi-annual coupon bond.

The response for this part is to be provided in the Excel spreadsheet.

The responses for all parts of this question are required on the paper provided to you.

6.

(*6 points*) Derivatives markets frequently exhibit volatility smiles where volatility varies with strike prices of options. The slope of the volatility smile cannot vary outside of certain bounds due to no-arbitrage constraints.

(a) (*1 point*) Explain why there should be an upper-bound on the slope of the volatility smile from call options.

(b) (*1 point*) Explain why there should be a lower-bound on the slope of volatility smile from put options.

For part (c), assume that:

- Volatilities are small.
- Strike, *K*, is either near or at-the-money forward.
- τ is time-to-expiration

(c) (*3 points*)

(i) Show that $-\sqrt{\frac{\pi}{2\tau}}\frac{1}{K}$ is approximately a no-arbitrage lower-bound on the slope of volatility smile from put option.

(ii) Show that $\sqrt{\frac{\pi}{2\tau}}\frac{1}{K}$ is approximately a no-arbitrage upper-bound on the slope of volatility smile from call options.

For part (d), assume that an equity index now trades at 5,000 and the Black-Scholes-Merton implied volatility for a three-month at-the-money European call option is 15%. Also assume no dividends and that the risk-free interest rate is zero.

(d) (*1 point*) Estimate the upper bound for implied volatility for three-month European calls with a strike of 5,050.

The responses for all parts of this question are required on the Excel provided to you.

7.

(*9 points*) Company PHX offers an investment product of 1-year maturity with a single premium of $1,000 at inception. The annual credited interest is based on the growth rate of the underlying equity index S over a 1-year period subjected to a minimum of 1% and a maximum (cap) of 5%, compounded annually with 100% participation.

Assume the following:

- Current underlying index value is 1,000
- Risk-free rate is 3%, continuously compounded
- Dividend yield = 0%
- Implied volatility = 15%
- One unit of equity option is traded on one unit of the underlying index; one can buy and sell fractional shares.
- 52 weeks per year
- All exposures are marked to market.

For part (a), assume that PHX takes a buy-and-hold strategy that replicates the exact positions embedded in this product offering. The premium collected is first used to fund the purchase of a 1-year risk-free zero-coupon bond to ensure the guaranteed minimum return at maturity. The remainder of the premium collected is then used to construct an option portfolio to replicate the annual credited interest.

For the current value of PHX's portfolio:

(a) (*3 points*)

(i) (*2 points*) Determine, for each equity option, the type (call/put), position (long/short), units, strike price, and the total current value.

The response for this part is to be provided in the Excel spreadsheet.

(ii) (*0.5 points*) Calculate the present value of the zero-coupon bond.

The response for this part is to be provided in the Excel spreadsheet.

(iii) (*0.5 points*) Determine the amount of the cash position, if any.

The response for this part is to be provided in the Excel spreadsheet.

7. Continued

For part (b), assume that PHX invests in the underlying index and constructs a collar to manage the equity risk exposure.

(b) (*3 points*)

(i) Calculate PHX’s cash position at inception, if any.

The response for this part is to be provided in the Excel spreadsheet.

(ii) Determine the equity delta of the portfolio at inception.

The response for this part is to be provided in the Excel spreadsheet.

For part (c), assume that PHX is considering adding a variation to this product which is different from the existing design: it offers a 10% cap, and instead of a 1% annual minimum, it offers a buffer that protects the policyholder’s initial investment from the first 10% loss. Assume PHX uses the collected premium to purchase a 1-year risk-free zero-coupon bond with the face amount of 1000 and the remainder to construct an option portfolio.

(c) (*2.5 points*) Analyze whether the premium covers the cost of a buy-and-hold strategy for the 10% cap and 10% buffer under the current market conditions.

The response for this part is to be provided in the Excel spreadsheet.

(d) (*0.5 points*) Explain how this new design can be appealing to the customers.

The response for this part is to be provided in the Excel spreadsheet.

The responses for some parts of this question are required on the paper provided to you.

8.

(*6 points*) The diagram below shows two plots (V1 and V2) of the implied volatility of an equity index at different strike prices for options with 6-month expirations. V1 is the current plot, and you believe V2 is what the volatility vs. strike plot should be.

Use the following assumptions for the questions below:

- Current index value $S = 100$
- Risk-free rate (continuously compounding) $r = 2\%$
- Three strike levels of interest: $K = 90, 100, 110$ always assuming 100 is the mid-point when applicable.
- Time to maturity = 0.5 years

Strike K	**90**	**100**	**110**
V1 (current) σ	24%	18%	16%
V2 (new) σ	19%	13%	11%

Your boss asks why options with lower strikes have higher implied volatility than those with higher strikes.

(a) (*1 point*) List two reasons that explain why.

8. **Continued**

You are going to construct a straddle to hedge the volatility risk.

(b) (*2.5 points*)

(i) (*1 point*) Describe the construction of the strategy; specify the long/short position.

(ii) (*1.5 points*) Plot the Vega as a function of the underlying index price using V2 volatilities.

The response for this part is to be provided in the Excel spreadsheet.

You are going to construct a strangle for volatility hedging.

(c) (*2.5 points*)

(i) (*1 point*) Describe the construction of the strategy; specify the long/short position.

(ii) (*1.5 points*) Plot the Vega as a function of the underlying index price with V2 volatilities.

The response for this part is to be provided in the Excel spreadsheet.

The responses for some parts of this question are required on the paper provided to you.

9.

(*9 points*) Your company is currently offering a T-year single premium Equity Indexed Annuity (EIA) with the crediting strategy of a classic point-to-point (PtP) design.

Payoff at Maturity T is given by $S_0 \max\left(\left(\frac{S_T}{S_0}\right)^{\alpha}, e^{gT}\right)$ where α is the participation rate, and g is the minimum guaranteed rate of return.

The price of the point-to-point design, *PtP(T)*, is given by

$$PtP(T) = S_0\left[e^{\left((\alpha-1)r+\frac{1}{2}\alpha(\alpha-1)\sigma^2\right)T}\Phi\left(\frac{\left(r-\frac{1}{2}\sigma^2+\alpha\sigma^2\right)T-\frac{g}{\alpha}T}{\sigma\sqrt{T}}\right)\right.$$
$$\left.+\,e^{(g-r)T}\Phi\left(\frac{\frac{g}{\alpha}T-\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right]$$

where Φ is the standard normal c.d.f.

(a) (*1 point*) Identify three underlying assumptions necessary for the above pricing formula to be valid.

For part (b), given that $r = 5\%$, $T = 5$, $\sigma = 20\%$, $g = 2\%$, $S_0 = 100$:

(b) (*2 points*)

(i) Calculate the prices of the point-to-point option for participation rates α of 60% and 120%.

The response for this part is to be provided in the Excel spreadsheet.

(ii) Estimate the participation rate of the point-to-point option such that the price is \$98.

The response for this part is to be provided in the Excel spreadsheet.

9. Continued

Your company would like to offer an EIA with a higher participation rate to make it a more attractive value proposition to customers/policyholders. The product development and pricing team think we can consider a T-year single premium Equity Indexed Annuity (EIA) with the crediting strategy of a double threshold design.

Specifically, the product credits the realized annualized returns on the index with a participation rate α_1 , as long as that is greater than the minimum guaranteed return 2%, and it does so only if the index annualized returns reach a certain threshold B_1.

It will credit a different participation rate α_2 provided that the index annualized returns reach a higher threshold B_2 (where $B_2 \geq B_1 \geq g,\ \ \frac{B_2}{\alpha_2} > \frac{B_1}{\alpha_1}$).

Specifically, the payoff at maturity T is:

$$
\begin{array}{ll}
S_0\,(\frac{S_T}{S_0})^{\alpha_2} & if\ S_T \geq S_0 e^{B_2 T/\alpha_2} \\
S_0\,(\frac{S_T}{S_0})^{\alpha_1} & if\ S_0 e^{B_1 T/\alpha_1} \leq S_T < \ S_0 e^{B_2 T/\alpha_2} \\
S_0 e^{gT} & if\ S_T < S_0 e^{B_1 T/\alpha_1}
\end{array}
$$

where B_1 , B_2 , α_1 , α_2 are positive constants less than 1, while $B_2 \geq B_1,\ \ \frac{B_2}{\alpha_2} > \frac{B_1}{\alpha_1}$.

The price of this double threshold design (DTD) is

$$DTD(T) = S_0 e^{-rT}\{\Omega_1 + \ \Omega_2 + \Omega_3\}$$

where

$$\Omega_1 = \mathbb{E}\left[e^{gT}\ \ \mathbb{I}_{\left\{\ln(S_T/S_0)\leq \frac{B_1}{\alpha_1}T\right\}}\right] = \ e^{gT}\Phi\left(\frac{-\left(r-\frac{1}{2}\sigma^2\right)T+\frac{B_1}{\alpha_1}T}{\sigma\sqrt{T}}\right)$$

$$\Omega_2 = \mathbb{E}\left[(\frac{S_T}{S_0})^{\alpha_2}\ \mathbb{I}_{\left\{\ln(S_T/S_0)> \frac{B_2}{\alpha_2}T\right\}}\right]$$

$$= e^{\alpha_2\left(r+\frac{1}{2}(\alpha_2-1)\sigma^2\right)T}\,\Phi\left(\frac{\left(r-\frac{1}{2}\sigma^2+\alpha_2\sigma^2\right)T-\frac{B_2}{\alpha_2}T}{\sigma\sqrt{T}}\right)$$

$$\Omega_3 = \mathbb{E}\left[(\frac{S_T}{S_0})^{\alpha_1}\ \mathbb{I}_{\left\{\frac{B_1}{\alpha_1}T<\ln(S_T/S_0)\leq \frac{B_2}{\alpha_2}T\right\}}\right] =$$

$$e^{\alpha_1\left(r+\frac{1}{2}(\alpha_1-1)\sigma^2\right)T}\left[\Phi\left(\frac{\left(r-\frac{1}{2}\sigma^2+\alpha_1\sigma^2\right)T-\frac{B_1}{\alpha_1}T}{\sigma\sqrt{T}}\right)-\Phi\left(\frac{\left(r-\frac{1}{2}\sigma^2+\alpha_1\sigma^2\right)T-\frac{B_2}{\alpha_2}T}{\sigma\sqrt{T}}\right)\right]$$

9. Continued

Underlying Asset – Current Price (S_0)	100
Dividend Yield (q)	0%
Implied Volatility (σ)	20%
Term (T)	5
Risk-Free Rate (r)	5%
Threshold 1 (B_1)	$0.5B_2$
Threshold 2 (B_2)	X
Participation rate 1 (α_1)	$0.9\,\alpha_2$
Participation rate 2 (α_2)	Y
Minimum guarantee rate (g) %	2%

Suppose X = B_2 = 14%.

(c) (*1.5 points*) Calculate Y, the participation rate 2 (α_2), such that the price of the double threshold design is equal to $98, using the *DTD(T)* formula above.

The response for this part is to be provided in the Excel spreadsheet.

For part (d), you are given two options:

- Point-to-point (PtP) option with α of 54.5%
- Double threshold design option with B_2 = 14% and α_2 = 71.5%

(Reminder: B_1=0.5B_2 and α_1=0.9α_2)

(d) (*4.5 points*)

(i) (*2 points*) Verify that the prices for both options are equal.

The response for this part is to be provided in the Excel spreadsheet.

(ii) (*2.5 points*) Calculate the payoff at maturity for both options for the following annualized index returns:

The response for this part is to be provided in the Excel spreadsheet.

9. Continued

Annualized Returns	PtP	Double Threshold
3%		
8%		
13%		
18%		
23%		
28%		
33%		

The responses for all parts of this question are required on the paper provided to you.

10.

(*8 points*) Consider the following two-factor Vasicek model:

$$dr_t = d\phi_{1,t} + d\phi_{2,t}$$

where

$$d\phi_{1,t} = \gamma_1^*(\vec{\phi}_1 - \phi_{1,t})dt + \sigma_1 dX_1$$
$$d\phi_{2,t} = -\gamma_2^* \phi_{2,t} dt + \sigma_2 dX_2$$

with

$$dX_1 dX_2 = \rho dt$$

(a) (*1 point*) Describe how this model incorporates dependency between the short rate r_t and the long-term yield rate $r_t(\tau)$ for suitable τ.

(b) (*2 points*) Describe how to simulate short rate paths using the transition density method.

You have two data sets, Dataset1 and Dataset2. Dataset1 contains two columns, each row corresponds to daily overnight rates and five-year zero-coupon yields over 505 consecutive trading days. There are 252 trading days a year. Dataset2 contains 28 yields for maturities ranging from 0.3 years to 10 years on the day corresponding to the last row of Dataset1. All the rates in both datasets are annualized, continuously compounded rates. The analysis carried out is given in the attached R output.

(c) (*3 points*) Explain the procedure used in calibrating the model.

(d) (*1.5 points*) Estimate the parameters $\gamma_1^*, \vec{\phi}_1, \sigma_1, \gamma_2^*, \sigma_2$ and ρ.

(e) (*0.5 points*) Assess the adequacy of the fit.

10. Continued

R Output

Codes for two functions

```
#'This function calculates log of  bond prices when the inputs are observed
#' short term volatility, long term volatility
#'
#'@param t current time in years.
#'@param T maturity value; the yield is measured at time  for a
#'zero-coupon bond that matures at T.
#'@param Tau time in years of long term rate.
#'@param rt current short term rate.
#'@param rlTau current long term rate.
#'@param gamma1,gamma2,phi1,phi2, are
#'\eqn{\gamma_1^*,\gamma_2^*,\phi_1} and \eqn{\phi_2},respectively.
#'@param short.term.vol,long.term.vol,correlation are volatility of short rate,
#'volatility of long rate and correlation between long rate and short  rate.

#' and correlation between short and long term rates.
#'@export
Two.factor.Vasicek.yield = function(t,T,Tau,rt,rlTau,phi1,phi2,gamma1,gamma2,
                              short.term.vol,long.term.vol,correlation){
  est = Two.factor.Vasicek.Vols(gamma1,gamma2,Tau,short.term.vol,
                              long.term.vol,correlation)
  sigma1= est[1]
  sigma2 = est[2]
  rho = est[3]
  B1Tau = Vasicek.B(gamma=gamma1,T=Tau)
  B2Tau = Vasicek.B(gamma=gamma2,T=Tau)
  ATau = Two.factor.Vasicek.A(phi1,phi2,gamma1,gamma2,sigma1,sigma2,rho,Tau)
  phi1t = (B2Tau*rt-Tau*rlTau-ATau)/(B2Tau-B1Tau)
  phi2t = (Tau*rlTau+ATau-rt*B1Tau)/(B2Tau-B1Tau)
  AtT = Two.factor.Vasicek.A(phi1,phi2,gamma1,gamma2,sigma1,sigma2,rho,(T-t))
  B1tT = Vasicek.B(gamma=gamma1,(T-t))
  B2tT = Vasicek.B(gamma=gamma2,(T-t))
  return ((-AtT+B1tT*phi1t+B2tT*phi2t)/(T-t))
}
```

10. Continued

```
#' This function calculates sigma1, sigma 2 and rho when volatility of short,
#' volatility of long rate and
#' correlation between long and short in equations are given.


#'@param gamma1,gamma2 are \eqn{\gamma_1^*} and \eqn{\gamma_2^*},respectively.
#'@param short.term.vol,long.term.vol,correlation are volatility of short rate,
#'volatility of long rate and correlation between long rate and short  rate.
#'@param Tau time in years of long term rate.
#'@export
Two.factor.Vasicek.Vols = function(gamma1=0.8269,
                                   gamma2=-0.0288,Tau=5,short.term.vol=0.0221,
                                   long.term.vol=0.0125,correlation=0.4713){
  B1Tau = Vasicek.B(gamma=gamma1,T=Tau)/Tau
  B2Tau = Vasicek.B(gamma=gamma2,T=Tau)/Tau
  if (correlation==0) {
    A = matrix(c(1,1,B1Tau^2,B2Tau^2),nrow=2,ncol=2,byrow=TRUE)
    B = matrix(c(short.term.vol^2,long.term.vol^2),nrow=2,ncol=1)
    est= solve(A,B)
    paras = c(est[1]^0.5,est[2]^0.5,0)
    return(paras)
  }
  else {
    cova = short.term.vol*long.term.vol*correlation
    A = matrix(c(1,1,2,B1Tau^2,B2Tau^2,2*B1Tau*B2Tau,B1Tau,B2Tau,(B1Tau+B2Tau))
           ,nrow=3,ncol=3,byrow=TRUE)
    B = matrix(c(short.term.vol^2,long.term.vol^2,cova),nrow=3,ncol=1)
    est= solve(A,B)
    paras = c(est[1]^0.5,est[2]^0.5,est[3]/(est[1]*est[2])^0.5)
    return(paras)
  }
}
```

10. Continued

Codes used for the anlysis

```
rm(list=ls()) # clear the work space and functions
graphics.off() # clear the plots

library(InterestCalibrationv1) # load the developed library
library(matrixStats) # load the library to calculate summary statistics
library(nlsr) # load the newest non-linear-least saure package
library(YieldCurve) # load the yield curve package.

## Loading required package: xts

## Loading required package: zoo

##
## Attaching package: 'zoo'

## The following objects are masked from 'package:base':
##
##     as.Date, as.Date.numeric

options(digits=5) # this for printing
rm(list=ls()) # clear the workspace and functions
load("Dataset1.Rda")
load("Dataset2.Rda")
head(Dataset1,n=3)

##   Short rate 5-Year Yield
## 1   0.030407     0.058691
## 2   0.032193     0.059278
## 3   0.031700     0.059372

tail(Dataset1,n=3)

##     Short rate 5-Year Yield
## 503   0.019152     0.047692
## 504   0.018500     0.047892
## 505   0.015000     0.045400

head(Dataset2,n=3)

##   Maturities    Yield
## 1    0.34795 0.028848
## 2    0.60000 0.035549
## 3    1.10411 0.044072

tail(Dataset2,n=3)

##    Maturities    Yield
## 26     8.1047 0.030906
## 27     8.6002 0.028260
## 28     9.1046 0.025561

Delta = 1/252
sigmar=sd(diff(Dataset1$`Short rate`))*(1/Delta)^0.5
sigmatau=sd(diff(Dataset1$`5-Year Yield`))*(1/Delta)^0.5
rho0=cor(diff(Dataset1$`Short rate`),diff(Dataset1$`5-Year Yield`))

Two.factor.Vasicek.fit =
```

10. Continued

```
  nls(Yield~Two.factor.Vasicek.yield(t=0,T=Maturities,
                                     Tau=5,rt=0.015,r1Tau=0.0454,
                                     phi1,phi2=0,gamma1,gamma2,
                                     short.term.vol = sigmar,
                                     long.term.vol = sigmatau,
                                     correlation = rho0),
      start=list(phi1=0.3,gamma1=1.6,gamma2=-0.02),
      data=Dataset2,algorithm = "port",
      lower=list(phi1=-2,gamma1=-2,gamma2=-2),
      upper =list(phi1=2,gamma1=2,gamma2=2),
      nls.control(maxiter = 10000, tol = 1e-8,
                  minFactor = 1/10240,
                  printEval = FALSE,
                  warnOnly = TRUE, scaleOffset = 0,
                  nDcentral = FALSE),trace=FALSE)
summary(Two.factor.Vasicek.fit)
```

```
##
## Formula: Yield ~ Two.factor.Vasicek.yield(t = 0, T = Maturities, Tau = 5,
##     rt = 0.015, r1Tau = 0.0454, phi1, phi2 = 0, gamma1, gamma2,
##     short.term.vol = sigmar, long.term.vol = sigmatau, correlation = rho0)
##
## Parameters:
##        Estimate Std. Error t value Pr(>|t|)
## phi1    0.28457    0.02422    11.8  1.1e-11 ***
## gamma1  1.74802    0.01692   103.3  < 2e-16 ***
## gamma2 -0.04090    0.00384   -10.6  9.0e-11 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 9.44e-05 on 25 degrees of freedom
##
## Algorithm "port", convergence message: relative convergence (4)
```

```
Two.factor.Vasicek.Vols(
  gamma1=as.numeric(coef(Two.factor.Vasicek.fit))[2],
  gamma2=as.numeric(coef(Two.factor.Vasicek.fit))[3],
  Tau=5,
  short.term.vol=sigmar,
  long.term.vol=sigmatau,
  correlation=rho0)
```

```
## [1]  0.020998  0.011652 -0.272098
```

****END OF EXAMINATION****