QFI Quantitative Finance Exam

Exam QFIQF

Date: Wednesday, April 28, 2021

INSTRUCTIONS TO CANDIDATES

General Instructions

1. This examination has 15 questions numbered 1 through 15 with a total of 100 points.

   The points for each question are indicated at the beginning of the question.

2. While every attempt is made to avoid defective questions, sometimes they do occur. If you believe a question is defective, the supervisor or proctor cannot give you any guidance beyond the instructions provided in this document.

3. Each question part or subpart should be answered either in the Word document, Excel document, or the paper provided as directed. Graders will only look at the work as indicated.

4. In the Word document, answers should be entered in the box marked ANSWER. The box will expand as lines of text are added. There is no need to use special characters or subscripts (though they may be used). For example, β_1 can be typed as beta_1 (and ^ used to indicate a superscript).

5. Prior to uploading your Word and Excel files, each file should be saved and renamed with your five-digit candidate number in the filename.

6. The Word file and the Excel file that contain your answers must be uploaded before time expires.

Written-Answer Instructions

1. Write your candidate number at the top of each sheet. Your name must not appear.

2. Write on only one side of a sheet. Start each question on a fresh sheet. On each sheet, write the number of the question that you are answering. Do not answer more than one question on a single sheet.

3. The answer should be confined to the question as set.

4. When you are asked to calculate, show all your work including any applicable formulas. When you are asked to recommend, provide proper justification supporting your recommendation.

5. When you finish, hand in all your written-answer sheets to the Prometric center staff. Be sure to hand in all your answer sheets because they cannot be accepted later.

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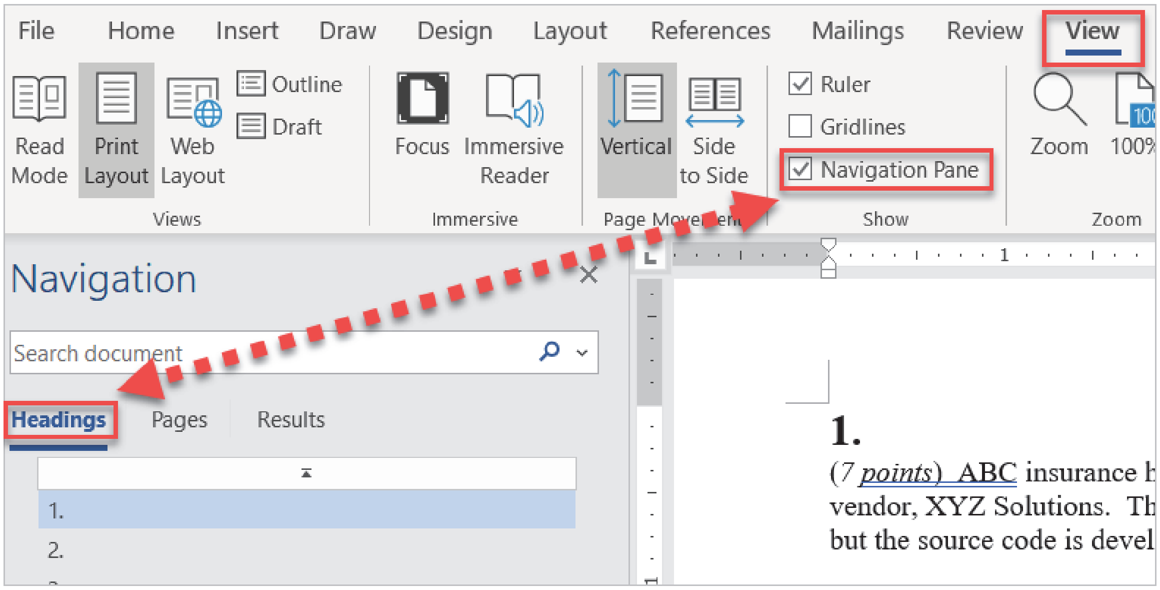
Navigation Instructions

Open the Navigation Pane to jump to questions.

Press Ctrl+F, or click View > Navigation Pane:



The responses for all parts of this question are required on the paper provided to you.

1.

(*6 points*) Under the real-world measure $\mathbb{P}$, the risk-free asset B_t and the stock price S_t satisfy the following processes:

$$dB_t = 0.01 B_t dt, \quad 0 \le t \le 1$$

$$dS_t = \begin{cases} 0.05 S_t dt + 0.2_t dW_t, & 0 \le t \le 0.5, \\ -0.05 S_t dt + 0.3 S_t dW_t, & 0.5 < t \le 1, \end{cases}$$

with $B_0 = S_0 = 1$, where $\{W_t : 0 \le t \le 1\}$ is a standard Brownian motion under $\mathbb{P}$. Let $\mathbb{Q}$ represent the risk-neutral measure.

(a) (*1 point*) Determine the market price of risk for all $t \le 1$.

(b) (*1 point*) Calculate $E^{\mathbb{P}}\left[S_1 | S_{0.5} \right]$.

(c) (*1 point*) Derive the Radon-Nikodym derivative of the risk-neutral measure $\mathbb{Q}$ with respect to the real-world measure $\mathbb{P}$.

(d) (*2 points*) Show that $\{S_t e^{-0.01t} : 0 \le t \le 1\}$ is a $\mathbb{Q}$-martingale.

The following are statements about risk-neutral pricing:

A. There is no arbitrage under the real-world measure if and only if there is no arbitrage under the risk-neutral measure.

B. The probability of an option ending up in the money does not depend on the real-world measure.

(e) (*1 point*) Comment on whether each statement is true or not.

The responses for all parts of this question are required on the paper provided to you.

2.

(*6 points*) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{W_t\}_{t\geq 0}$ be a standard Brownian motion with respect to the filtration $\{\mathcal{F}_t\}_{t\geq o}$.

(a) (*0.5 points*) List the criteria for the stochastic process V_t to be a sub-martingale with respect to $(\Omega, \mathcal{F}, \mathbb{P})$.

(b) (*2 points*) Evaluate $Var^{\mathbb{P}}\left[|W_t|\right]$.

You are given the conditional version of Jensen's inequality, which states that for a convex function $\psi(X)$ and integrable random variable X:

$$E^{\mathbb{P}}\left(\left(\psi(X)\middle|\mathcal{F}_t\right)\right) \geq \psi\left(E^{\mathbb{P}}\left(X\middle|\mathcal{F}_t\right)\right)$$

(c) (*2 points*) Prove that $|W_t|$ is a non-negative sub-martingale.

(d) (*1.5 points*) Determine integer *k* that makes W_t^k a martingale.

The responses for all parts of this question are required on the paper provided to you.

3.

(*7 points*) Let W_t be a standard Wiener process defined on the interval $[0, T]$ with $\{\mathcal{F}_t\}_{0 \le t \le T}$ its natural filtration.

You are given that for a standard normal random variable *Z*, we have $E\left[Z^4\right] = 3$.

(a) (*2 points*) Derive $E\left[W_s^3 W_t\right]$ for $t > s$.

(b) (*1.5 points*) Determine the value of *c* such that $W_t^3 - ctW_t$ is a martingale.

(c) (*2 points*) Show that $X_t = \int_0^t W_u du$ is not a martingale.

Let $V = \int_0^1 e^{-s} dW_s$ and $Y = \int_0^2 e^{-s} dW_s$.

(d) (*1.5 points*) Calculate

(i) $E[V^2]$

(ii) $E[VY]$

The responses for all parts of this question are required on the paper provided to you.

4.

(*5 points*) Assume that the price S_t at time $t \geq 0$ of a non-dividend paying stock follows the stochastic differential equation below:

$$\frac{dS_t}{S_t} = 0.045\, dt + \sigma\, dW_t$$

where σ is a positive constant and W_t is a standard Wiener process under the risk-neutral probability measure.

You are given that there is a real number c such that:

$$\frac{d\left(S_t\right)^c}{\left(S_t\right)^c} = 0.18\, dt + 0.6\, dW_t$$

(a) (*2.5 points*) Show, using Ito's lemma, that $\sigma = 0.3$.

Recall that for any real number k and a standard normal random variable Z:

$$E\left[Z^2 e^{kZ}\right] = \left(1 + k^2\right) e^{0.5k^2}$$

Consider a derivative security on the stock. The derivative security pays $S_3\left[\ln S_3\right]^2$ at time 3, and nothing at any other time. Assume $S_0 = 1$.

(b) (*2.5 points*) Calculate the time-0 no-arbitrage price of this derivative security.

5.

(*8 points*)

(a) (*2 points*) Compare interest rate futures and forwards, and discuss the advantages/disadvantages of futures compared to forwards.

ANSWER:

You are given the interest rate (continuous compounding) at time 0 and year 1 below, and asked to perform analysis of a forward contract on a 5-year note with the annual coupon rate of 4% and maturity in 2 years.

Tenor	time 0	year 1
1 Yr	2.63%	0.16%
2 Yr	2.48%	0.16%
3 Yr	2.46%	0.18%
4 Yr	2.51%	0.29%
5 Yr	2.51%	0.29%
6 Yr	2.59%	0.49%
7 Yr	2.59%	0.49%
10 Yr	2.69%	0.66%

(b) (*1.5 points*) Compute the value of the forward contract at time 0 and year 1.

The response for this part is to be provided in the Excel spreadsheet.

Yesterday you bought a 5-year Treasury note future expiring in 2 years and today the future price drops to $100. The future contract size is $1,000,000. Your broker requires initial Margin: $1,485 (per contract); Maintenance Margin: $1,110 (per contract).

(c) (*1 point*) Calculate the cash flow today.

The response for this part is to be provided in the Excel spreadsheet.

5. Continued

You are given the following data at time 0:

- The 6-month zero coupon bond is priced at $98.24
- The 9-month zero coupon bond is priced at $97.21
- Call option (European) on the 3-month Treasury bill with maturity in 6 months and strike price of $99.12 is priced at $0.2934
- Put option (European) on the 3-month Treasury bill with maturity in 6 months and strike price of $99.12 is priced at $0.2044

(d) (*1 point*) Explain why the above securities are priced incorrectly.

The response for this part is to be provided in the Excel spreadsheet.

(e) (*2.5 points*) Describe a strategy to take advantage of the arbitrage opportunity:

(i) if the 3-month Treasury bill price is higher than the strike price.

ANSWER:

(ii) if the 3-month Treasury bill price is lower than the strike price.

ANSWER:

6.

(*7 points*) Your company bought a 5-year 6% coupon bond paid semi-annually with par value of 100 million.

Assume that the current term structure of the interest rate is flat at continuously compounded rate of 3.5%.

Period i	T_i	Cash Flow CF (millions)	Discount $Z(0, T_i)$	Discounted CF	Weight w_i	Weight x T_i	Weight x T_i^2
1	0.5	3	0.9827	2.95	0.027	0.0133	0.0066
2	1.0	3	0.9656	2.90	0.026	0.0260	0.0260
3	1.5	3	0.9489	2.85	0.026	0.0384	0.0576
4	2.0	3	0.9324	2.80	0.025	0.0503	0.1006
5	2.5	3	0.9162	2.75	0.025	0.0618	0.1545
6	3.0	3	0.9003	2.70	0.024	0.0729	0.2186
7	3.5	3	0.8847	2.65	0.024	0.0835	0.2923
8	4.0	3	0.8694	2.61	0.023	0.0938	0.3752
9	4.5	3	0.8543	2.56	0.023	0.1037	0.4666
10	5.0	103	0.8395	86.46	0.777	3.8868	19.4341

The Chief Investment Officer (CIO) is worried about the losses its portfolio may suffer from an upward shift in the term structure of interest rates.

Analyst A proposes a duration hedging strategy against a parallel shift in the term structure of the interest rate by entering into a position of *k* units 5-year zero-coupon bond with par value of 100 million.

(a) (*1 point*) Calculate the value of *k* based on Analyst A's proposal.

The response for this part is to be provided in the Excel spreadsheet.

You are given the following additional information:

Yield (continuously compounded)	3.20%	3.40%	3.60%	3.80%	4.00%
Price of coupon bond ($ million)	112.72	111.72	110.74	109.76	108.79
Z (0,2)	0.9380	0.9343	0.9305	0.9268	0.9231
Z (0,5)	0.8521	0.8437	0.8353	0.8270	0.8187

(b) (*1 point*) Calculate the DV01 of the portfolio consisting of the original bonds plus hedging strategy calculated in part (a).

The response for this part is to be provided in the Excel spreadsheet.

6. **Continued**

The CIO is also concerned about a large shift in interest rate.

Analyst B proposes a duration and convexity hedging strategy against both small and large changes in interest rates by entering into a position of k_1 2-year zero-coupon bond and k_2 5-year zero-coupon bond.

(c) (*1.5 points*) Construct a hedging portfolio based on Analyst B's proposal.

The response for this part is to be provided in the Excel spreadsheet.

(d) (*1 point*) Identify two other hedging instruments (in addition to zero-coupon bonds) that your company can use to mitigate the risk of an upward shift in interest rates.

ANSWER:

After the purchase, the CIO becomes worried about a non-parallel shift in the term structure of interest rates. The sensitivity of interest rates to level, slope, and curvature were shown in the table below.

Analyst C proposes a factor neutrality hedging strategy with 2-year zero-coupon bond S and 7-year zero-coupon bond L.

Factor (i)	1 year	2 year	3 year	5 year	7 year
β_{i1} (Level)	0.9196	1.0344	1.0299	1.0180	1.0111
β_{i2} (Slope)	-0.4317	-0.3507	-0.1424	0.2432	0.5205
β_{i3} (Curv.)	0.0847	0.3228	0.3240	0.1716	-0.1058
R^2	98.88%	99.61%	99.77%	99.90%	99.73%

(e) (*1 point*) Describe the procedure to construct a factor neutral hedge.

ANSWER:

(f) (*1.5 points*) Calculate factor duration of bond S and bond L by level, slope, and curvature.

The response for this part is to be provided in the Excel spreadsheet.

The responses for all parts of this question are required on the paper provided to you.

7.

(*5 points*) The short rate *r(t)* follows the following process:

$$dr_t = [v - ar_t]dt + \sigma dX_t$$

where X_t is a standard Brownian motion and v, a, and σ are positive constants.

(a) (*1.5 points*)

(i) Solve the stochastic differential equation.

(ii) Identify the distribution of r_t by providing its mean and variance.

(b) (*1 point*) Show that the limiting distribution of r_t as t approaches infinity is $N\left(\frac{v}{a}, \frac{\sigma^2}{2a}\right)$

Assume that $r_m \sim N\left(\frac{v}{a}, \frac{\sigma^2}{2a}\right)$ for some $m > 0$.

(c) (*1 point*) Demonstrate that the interest rate, r_{t+m}, follows the same distribution. Hint: Use time frame $(m, t+m)$ from solution of part (a).

(d) (*1.5 points*)

(i) Estimate the parameters for interest rate process above.

(ii) Describe for the estimation of arbitrage free parameters using the table below observed in the market.

Hint: In a linear regression on time series $x_i = a + \beta x_{i-1}$, the coefficients are

$$\beta = \frac{n \cdot \sum_{i=1}^{n} x_{i-1} \cdot x_i - \sum_{i=1}^{n} x_i \cdot \sum_{i=1}^{n} x_{i-1}}{n \cdot \sum_{i=1}^{n} x_{i-1}^2 - \left(\sum_{i=1}^{n} x_{i-1}\right)^2}$$

$$a = \frac{\left(\sum_{i=1}^{n} x_i - \beta \sum_{i=1}^{n} x_{i-1}\right)}{n}$$

7. **Continued**

i	t	r_i
0	0	0.03
1	0.25	0.034735
2	0.5	0.015108
3	0.75	0.049016
4	1	0.035745
5	1.25	0.03859
6	1.5	0.070412
7	1.75	0.037342
8	2	0.016796
9	2.25	0.057236
10	2.5	0.027911
11	2.75	0.044456
12	3	0.046928
13	3.25	0.06747
14	3.5	0.07093
15	3.75	0.040857
16	4	0.031479
17	4.25	0.042559
18	4.5	0.050222
19	4.75	0.026584
20	5	0.016775

$\sum_{i=1}^{20} r_i = 0.821151$

$\sum_{i=1}^{20} r_{i-1} = 0.834376$

$\sum_{i=1}^{20} r_i^2 = 0.039037$

$\sum_{i=1}^{20} r_{i-1}^2 = 0.03965$

$\sum_{i=1}^{20} r_{i-1} \cdot r_i = 0.034693$

$$Var(r)$$

$$= \frac{1}{20}\sum_{i=1}^{20} r_i^2 - \left(\frac{1}{20}\sum_{i=1}^{20} r_i\right)^2$$

The responses for all parts of this question are required on the paper provided to you.

8.

(*8 points*) Let A_t be price at time t of a traded asset, so that

$$\left\{B_t^{-1}A_t, t \geq 0\right\}$$

is a martingale under the measure $\mathbb{P}$ where B_t is a money market account. Let $F_A(t,T)$ be its forward price at t for settlement at T. Then $F_A(t,T) = \frac{A_t}{Z(t,T)}$, where $Z(t,T)$ is a zero-coupon bond value at t maturing at T.

(a) (*1.5 points*)

(i) Show that the forward price can also be computed as $F_A(t,T) = E_f^*\left(A_T \middle| \mathcal{F}_t\right)$ under the *T*-forward risk-neutral measure $\mathbb{Q}^T$.

(ii) Prove, using part (a)(i), that $\left\{F_A(t,T), t \geq 0\right\}$ is a martingale under the *T*-forward measure $\mathbb{Q}^T$.

The payoff of a put option in 9 months is given by:

$$g = \$100\ million \times \max\left(0.03^{1/3} - r(0.5, 0.75)^{1/3}, 0\right),$$

where $r(0.5, 0.75)$ is the 3-month LIBOR.

The current forward rate $f(0, 0.5, 0.75) = 3.0250\%$ and the current floorlet volatility $\sigma_f^{fwd}(0.75) = 20\%$. The zero-coupon bond price maturing in 9 months is 98.45.

(b) (*2 points*) Calculate the value of this put option.

Hint: For a log-normally distributed variable x,

$$E\left[x^a\right] = E\left[x\right]^a e^{\frac{(a-1)a}{2}\sigma_{\log(x)}^2}, Var\left[\log x^a\right] = a^2 \cdot \sigma_{\log(x)}^2$$

8. **Continued**

Negative interest rates are observed when the LIBOR Market Model (LMM) is considered as an option to evaluate interest sensitive securities. The LMM produces strict positive rates.

(c) (*1 point*) State two arguments to explain why low or negative interest rates were observed in 2008-2016 period in major markets.

(d) (*2.5 points*) Consider volatilities in a low interest rate environment and provide answers for following.

(i) (*1 point*) Describe the problem of using Black implied volatilities in interest rate sensitive derivative pricing and explain the advantages of using absolute/normal volatilities by describing practitioners' view.

(ii) (*0.5 points*) Describe the correlations between yield curve levels and volatilities for Black volatilities and absolute/normal volatilities.

(iii) (*1 point*) Recommend the choice of volatilities in the current low interest rate environment. Justify your recommendation.

(e) (*1 point*) Describe two approaches to adapt the lognormal forward diffusion LMM to accommodate negative forward rates.

The responses for all parts of this question are required on the paper provided to you.

9.

(*7 points*) Your firm has been using the G2++ 2-factor model for risk-managing its interest rate products. Under the G2++, the dynamics of the instantaneous short-rate process under $\mathbb{Q}$ is given by:

$$r(t) = x(t) + y(t) + \varphi(t), \; r(0) = r_0,$$

where the processes $\{x(t): t \geq 0\}$ and $\{y(t): t \geq 0\}$ satisfy:

$$dx(t) = -ax(t)dt + \sigma dW_1(t), \; x(0) = 0$$
$$dy(t) = -by(t)dt + \eta dW_2(t), \; y(0) = 0$$

with W_1 and W_2 two correlated standard Brownian motion processes with instantaneous correlation, ρ, such that $dW_1(t)dW_2(t) = \rho dt$, and a, b, σ, and η are positive constants.

(a) (*2 points*) Derive an expression for $r(t)$.

(b) (*2 points*) Derive expressions for $E\{r(t)\}$ and $Var\{r(t)\}$.

Your colleague has read that, to calibrate the model, the following must be true and asks how the calibration is performed in practice.

$$\varphi(T) = f^M(0,T) + \frac{\sigma^2}{2a^2}\left(1 - e^{-aT}\right)^2 + \frac{\eta^2}{2b^2}\left(1 - e^{-bT}\right)^2$$
$$+\rho\frac{\sigma\eta}{ab}\left(1 - e^{-aT}\right)\left(1 - e^{-bT}\right)$$

where $f^M(0,T)$ is the instantaneous forward rates at time 0.

(c) (*1 point*) Explain the steps involved in the calibration process for the G2++ model in practice.

9. Continued

Your colleague appreciates your explanation and asks how likely it is that in the current low-rate environment, the G2++ produces negative instantaneous rates and whether the model can support them.

(d) (*1 point*) Derive an expression for the risk-neutral probability that the instantaneous rate $r(t)$ is negative.

(e) (*0.5 points*) Comment on the pros and cons of using the G2++ model

(f) (*0.5 points*) Comment on how appropriate this model would be for modeling guarantees that are significantly out of the money.

The responses for all parts of this question are required on the paper provided to you.

10.

(*7 points*) Consider a Vasicek model of interest rates as follows:

$$dr_t = \gamma(\bar{r} - r_t)dt + \sigma dX_t,$$
$$m(r,t) = \gamma(\bar{r} - r_t)$$

And the arbitrage-free condition for long-short bond portfolio with Z_1, Z_2 is shown

$$\frac{\frac{\partial Z_1}{\partial t} + \frac{1}{2}\frac{\partial^2 Z_1}{\partial r_t^2} - r_t Z_1}{\partial Z_1 / \partial r_t} = \frac{\frac{\partial Z_2}{\partial t} + \frac{1}{2}\frac{1\partial^2 Z_2}{\partial r_t^2} - r_t Z_2}{\partial Z_2 / \partial r_t} = -m^*(r,t)$$

(a) (*1 point*) Compare $m(r,t)$ with an arbitrage-free parameter $m^*(r,t)$ and explain the meaning of the parameters when $m^*(r,t) = \gamma^*(\bar{r}^* - r)$.

Consider the following parameters estimated as follows:

$$\gamma = 0.3262; \bar{r} = 5.09\%; \sigma = 2.21\%; \gamma^* = 0.4653; \bar{r}^* = 6.34\%$$

Assume today's short-term interest rate is $r_0 = 2\%$ and the bond pricing formula of Vasicek model

$$Z(r,t;T) = e^{A(t;T) - B(t;T)r}$$

where $B(t;T) = \frac{1}{\gamma^*}\left(1 - e^{-\gamma^*(T-t)}\right)$

and $A(t;T) = B(t;T) - (T-t)\left(\bar{r}^* - \frac{\sigma^2}{2\gamma^*}\right) - \frac{\sigma^2 B(t;T)^2}{4\gamma^*}$

(b) (*3 points*) Show that $E\left[\frac{dZ/dt}{Z}\right] = E(r_t) + \frac{\sigma^2 B}{2\gamma^*}(1 - \gamma^*)$ using Ito's lemma.

10. Continued

(c) (*1 point*) Compute $E\left[\frac{dZ/dt}{Z}\right]$ on zero-coupon bond with 10 year to maturity.

You are given $Z(r_0, 0; 1) = 0.975$ and $Z(r_0, 0; 5) = 0.898$.

(d) (*2 points*) Calculate the value of a call option with 1 year to maturity $(T_0 = 1)$, strike price $K = 0.9$, written on a zero-coupon bond with 5 years to maturity.

The responses for all parts of this question are required on the paper provided to you.

11.

(*8 points*) Assume that the underlying asset S_t under risk-neutral follows the Heston model:

$$\begin{cases} \dfrac{dS_t}{S_t} = rdt + \sqrt{v_t}\,dW_t^1 \\ dv_t = k\left(\theta^2 - v_t\right)dt + \gamma\sqrt{v_t}\,dW_t^2 \end{cases}$$

where r is the risk-free interest rate, k, θ^2 and γ are constant, and W_t^1 and W_t^2 are Wiener processes correlated by ρ.

A self-financing portfolio Π_t is constructed as $\Pi_t = \zeta U_t + \eta V_t + \theta S_t$ where U_t and V_t are derivatives on S_t and ζ, η, θ are dynamic weights such that

$$d\Pi_t = \zeta dU_t + \eta dV_t + \theta dS_t$$

(a) (*3 points*)

(i) Derive $d\Pi_t$ in terms of $dt, dS_t,$ and dv_t.

(ii) Construct a riskless portfolio Π_t to show that

$$\frac{F(V_t) - rV_t + rS_t \dfrac{\partial V_t}{\partial S_t}}{\dfrac{\partial V_t}{\partial v_t}} = \frac{F(U_t) - rU_t + rS_t \dfrac{\partial U_t}{\partial S_t}}{\dfrac{\partial U_t}{\partial v_t}}$$

where $F(X_t) = \dfrac{\partial X_t}{\partial t} + \dfrac{1}{2} v_t S_t^2 \dfrac{\partial^2 X_t}{\partial S_t^2} + \dfrac{1}{2}\gamma^2 v_t \dfrac{\partial^2 X_t}{\partial v_t^2} + \gamma v_t \rho S_t \dfrac{\partial^2 X_t}{\partial S_t \partial v_t}$ for any derivative X_t on S_t.

11. Continued

Denote by $f\left(S_t, v_t, t\right) = \dfrac{F\left(V_t\right) - rV_t + rS_t \dfrac{\partial V_t}{\partial S_t}}{\dfrac{\partial V_t}{\partial v_t}}$ which is the same for all derivatives V_t on S_t based on the result in part (a)(ii).

(b) (*0.5 points*) Derive the generic partial differential equation that any derivative V_t on S_t must follow.

Given the following data:

r	k	θ	γ	ρ	S_0	v_0	z_1	z_2
0%	0.7	0.4	0.3	0.1	100	0.09	-0.7	0.9

z_1 and z_2 are two independent simulations from a standard normal distribution.

(c) (*1.5 points*) Calculate the simulated value of S_t at $t = 0.04$.

(d) (*1 point*) Describe how to produce the volatility smile implied by this Heston model.

Given that the distribution of S_t produced by the Heston model has fatter tails than a log-normal distribution,

(e) (*2 points*)

(i) Explain why the Heston model would produce a volatility smile.

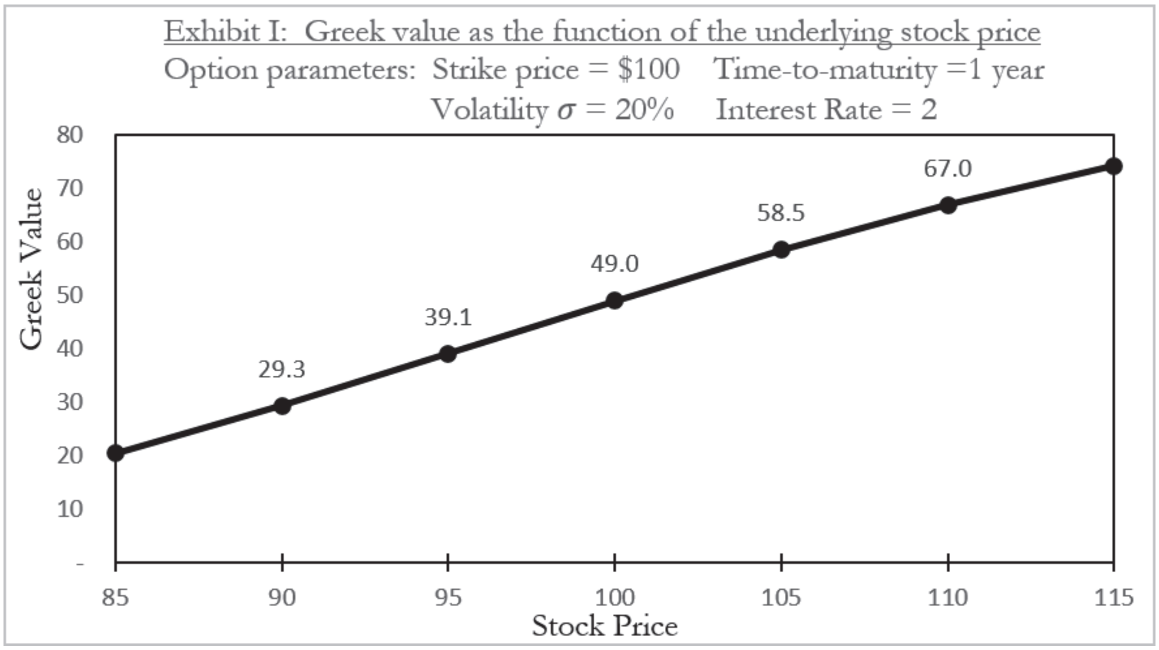
(ii) Describe one way to calibrate the Heston model.

(iii) Describe two potential disadvantages of such calibration.

The responses for all parts of this question are required on the paper provided to you.

12.

(*7 points*) Exhibit I below shows the Greek of a European call option on a non-dividend-paying stock based on the Black-Scholes-Merton (BSM) model.



(a) (*0.5 points*) Determine which Greek (Delta, Gamma, Vega, Rho, or Theta) Exhibit I represents. Justify your answer. (Here Theta is defined as the derivative of the option value with respect to the passage of time.)

(b) (*2 points*) Draw "Line A" in Exhibit I to show the same Greek of a European put option that has the same parameters as the one in Exhibit I. Indicate the Greek value in "Line A" at stock price = 100. You need not show other values in "Line A" but comment on the slope of this line.

(c) (*2 points*) Draw "Line B" in Exhibit I to show the same greek of a European put option that has the same parameters as in Exhibit I, except that the time-to-maturity is 1 month. Indicate the Greek value in "Line B" at stock price = 85. You need not show other values in "Line B" but comment on the slope of this line.

12. Continued

Exhibit II below shows Vega and Gamma for a European option on a non-dividend-paying stock. These Greek values are derived from the BSM model with the same strike price, volatility, interest rate, and time-to-maturity as in Exhibit I.

Exhibit II: Vega and Gamma with respect to the underlying stock price

Stock price	60	X
Vega (shown as the change in the option value to 1 percentage point change of the volatility, e.g., from 25% to 26%)	0.2401	0.2548
Gamma	0.0267	0.0159

(d) (*1.5 points*) Determine the stock price X in Exhibit II.

Assume that the time-to-maturity that is used to construct Exhibit II exceeds 1 year.

(e) (*1 point*) Determine an upper bound of the option's implied volatility.

The responses for all parts of this question are required on the paper provided to you, except part (c)(i) and (ii).

13.

(*8 points*) You are an actuary working on an investment product, where the interest credited is based on the growth rate of an underlying stock index S over a 1-year period, and crediting parameters including cap rate c and participation rate p. The interest credited will be floored at a non-negative guaranteed rate g.

The interest crediting strategy for this product per \$1 at time t is specified as:

$$Interest\ Credited_t = \max\left\{\min\left[\left(\frac{S_t}{S_{t-1}} - 1\right)\cdot p, c\right], g\right\}$$

Given the following data:

- Cap rate c is 5%
- Participation rate p is 90%
- Guaranteed rate g is 1%
- Continuous risk-free rate r is 5%
- Implied volatility σ is 20%

(a) (*2.5 points*) Derive the replicating portfolio using options for the interest credited above the guaranteed rate, i.e. $Interest\ Credited_t - g$. Specify each option, including position, option type, term, and strike ratio K / S_{t-1}.

(b) (*1 point*) Sketch the payoff of the replicating portfolio against the index growth rate $\left(\frac{S_t}{S_{t-1}} - 1\right)$.

13. Continued

Given the interest crediting period is from Dec 31, 2018 to Dec 31, 2019, index value $S_{t-1} = 1000$ on Dec 31, 2018, and $S_t = 1080$ on Dec 31, 2019, respectively. Assume 1000 of initial investment:

(c) (*2.5 points*)

(i) (*0.5 points*) Calculate the interest credited on Dec 31, 2019.

(ii) (*2 points*) Calculate the cost of the replicating portfolio for the interest credited above the guaranteed rate on Dec 31, 2018.

The response for this part is to be provided in the Excel spreadsheet.

Assume the implied volatility σ is in absence of transaction costs. Given the transaction cost per share of stock k is 0.52% and the replicating portfolio is rebalanced every week.

(d) (*1 point*)

(i) Calculate the effective volatility $\tilde{\sigma}$ that covers the transaction costs for long and short option positions, respectively. Assume 52 weeks per year and $\pi = 3.14$.

(ii) Justify the calculation of effective volatility regarding to each option position.

Regarding to rebalancing:

(e) (*1 point*)

(i) Describe the relationship between hedging frequency and the profit.

(ii) Describe strategies that can be used for rebalancing.

14.

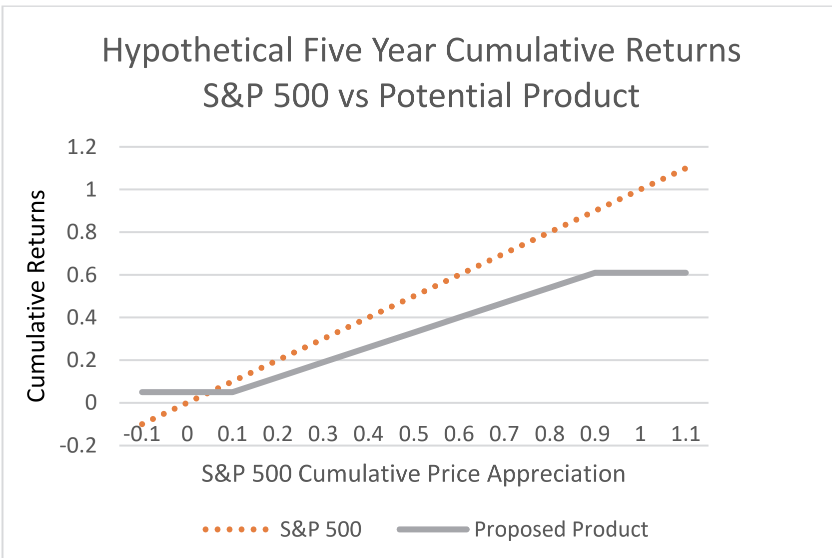
(*5 points*) You are a new actuarial assistant who recently joined a large international insurance company XYZ that has a large block of equity-index annuity (EIA) business.

In a recent research project, your manager asked you to research EIA contracts. Your manager wants to know more about common elements of an EIA contract, for example, the market-value adjustment which is in the formula expressed below, for time *t*:

$$MV\ Adj_t = \frac{(1+declared\ rate)^{T-t}}{(1+market\ rate)^{T-t}},$$

where T = original term-to-maturity of the EIA.

The payoff of a competitor's product can be found in the graph below. The product has a five-year term-to-maturity. Surrender charges must be paid if the product is not held to maturity.



(a) (*1 point*)

(i) Explain the features of the competitor's product.

ANSWER:

(ii) Describe attributes of the buyer of the competitor's product.

ANSWER:

14. Continued

(b) (*0.5 points*) Describe a condition when the market-value adjustment works in an investor's favor.

ANSWER:

Consider the following 1-year point-to-point EIA contract:

Initial investment	100,000
Index participation rate	75%
Minimum guarantee	3% interest rate credit on 100% of the initial investment
Yield-spread fee	1%

Now suppose that

- Yield of a 1-year zero-coupon bond is 3%
- 1-year implied volatility is 20%

(c) (*0.5 points*) Calculate the risk budget of the EIA contract.

ANSWER:

You are given the following 1-year European call option prices:

	Strike Price			
	0.90	0.97	1.00	1.03
Call option price	0.15429	0.11015	0.09413	0.07986

(d) (*1 point*) Assess if the risk budget in part (c) is sufficient to fund the option purchase that would replicate the EIA payoff in excess of the minimum return.

ANSWER:

(e) (*0.5 points*) Calculate the index participation rate to break-even using the risk budget calculated in part (c).

ANSWER:

14. Continued

Your manager made the following statements about EIA:

A. EIA contracts make money for an insurer if actual investment and mortality experience are better than the assumptions embedded in the product's actuarial reserve account.
B. The participation rate of an EIA is applied to the reference index's price appreciation only.
C. An EIA is a precautionary savings vehicle for potential future expenses during times of economic distress.

(f) (*1.5 points*) State and explain whether each of the above statements is true or false.

ANSWER:

The responses for all parts of this question are required on the paper provided to you.

15.

(*6 points*) Company JCP sells variable annuities (VA) with a GMWB rider. They are interested in implementing a hedging program.

(a) (*1 point*) Compare using a daily vs. less frequent rebalancing strategy for hedging VA guarantees.

Company JCP would like to study the effectiveness of different hedging strategies in realistic market environments featuring both stochastic volatility and stochastic interest rates.

Company JCP received statistical information below to evaluate the effectiveness of different hedging strategies. The data can be used to analyze how different hedging strategies would perform under three financial market models.

- Financial Market Models

Model	Equity Model	Interest Rate Model
1	Black – Scholes (BS)	One-factor CIR
2	BS	Three-factor CIR
3	Heston	Three-factor CIR

- Conditional Tail Expectation (CTE) 95% of an Insurer's Hedged loss at maturity

Hedging Strategies			**Financial Market Model**		
Hedging Greeks	**Rebalancing Frequency**	**Hedging Model**	**Model 1**	**Model 2**	**Model 3**
Δ	Monthly	BS	13	22	57
Δ- ρ	Monthly	BS	6	10	43
Δ- ρ	Daily	BS	3	6	35
Δ- ρ	Daily	BS-Vasicek (BSV)	2	8	34

Note: Hedged loss at maturity is defined as the insurer's unhedged loss on the VA contract less the terminal value of the hedging portfolio.

15. **Continued**

(b) (*2.5 points*)

(i) Compare the hedging strategies considered above (for the GMWB rider) with respect to the following aspects:

I. Hedging Delta (Δ) versus Delta - Rho (Δ- ρ)
II. Monthly versus daily rebalancing
III. BS versus BSV hedging model

(ii) Recommend which one of the four hedging strategies above that Company JCP should implement, based on part (b)(i).

(iii) Describe how the hedging strategy that you recommended would perform in a persistently low interest rate environment.

Company JCP would like to use a hedging model with stochastic volatility.

(c) (*1 point*) Define the two schools of thought regarding the calibration of the equity stochastic volatility parameter and their application to VA hedging programs.

Company JCP would like to calibrate their stochastic equity volatility parameter to the VIX.

(d) (*1.5 points*) Calculate the 1-year VIX_t under the following stochastic differential equation:

- $dv_t = 3(2 - v_t)\,dt + \sigma_v \sqrt{v_t}\,dW_t^v$
- $\lambda_v = -1$
- $\{W_t^v, t \geq 0\}$ is a standard Brownian motion under the real-world measure

****END OF EXAMINATION****